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Group classification and symmetry reduction of variable coefficient nonlinear diffusion–convection equation

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Abstract
Based on classical Lie group method, the group for a general class of variable coefficient nonlinear diffusion–convection equation in $(1+1)$ dimensions is obtained. New symmetries are found. Seven models have been studied.

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1. Introduction

The importance of the variable coefficient nonlinear diffusion–convection equation

$$f(x)u_t = (g(x)D(u)u_x)_x - k'(u)u_x \tag{1}$$

where

$$f(x), g(x) \neq 0 \quad \text{and} \quad D(u) \neq \text{const}$$

is well known and there is a continuing high level of interest in the construction of exact solutions to special forms of (1) [1–4]. In equation (1), the first term on the right-hand side describes diffusion with a generally non-constant diffusion, $D(u)$, whereas the second term describes convection, $k(u)$. Equation (1) has a wide range of applications in physical and related sciences. It describes, for the case $f(x) = g(x) = 1$, the vertical one-dimensional transport of water in homogeneous, non-deformable porous media. Also it describes for the case $k(u) = \text{const}$ many apparently unrelated phenomena, such as heat conduction in solids and stationary motion of a boundary layer of fluid over a flat plate [5, 6].

The functions $f(x)$ and $g(x)$ take into account some of the physical phenomena ignored in the general nonlinear diffusion–convection equations [1, 2, 7]. These include propagation of a nonlinear thermal wave in an inhomogeneous medium [6]. The most extensive table of symmetries for the nonlinear diffusion–convection equation, for the case $f(x) = g(x) = 1$, is given in [7] and later in [1, 2] extra symmetries are found.

On using the following transformations

$$\breve{x} = \int f(x)\,\mathrm{d}x \qquad \breve{t} = t$$

equation (1) reduces to

$$\frac{\partial u}{\partial \breve{t}} = \frac{\partial}{\partial \breve{x}}\left(fgD(u)\frac{\partial u}{\partial \breve{x}}\right) - k'\frac{\partial u}{\partial \breve{x}}. \tag{2}$$

Now the two arbitrary functions f and g can be combined into a single new function. Hence equation (1) has only three free parameters instead of four. Nevertheless, here we prefer to deal with equation (1) rather than equation (2) according to several models which can easily be constructed and classified in terms of $f(x)$, $g(x)$, $D(u)$ and $k(u)$.

The aim of the present paper is to identify all classes of the variable coefficient nonlinear diffusion–convection (VCNDC) equation that have nontrivial symmetry group.

2. Lie group symmetry analysis

In the present paper, classical Lie group theory [5, 8–10] is used to determine the classical symmetries of equation (1). The conditions that the arbitrary functions $f(x)$, $g(x)$, $D(u)$ and $k(u)$ have to fulfil for equation (1) to admit the symmetries have been achieved. These symmetries are obtained by considering the infinitesimal transformation

$$u^* = u + \epsilon\eta(x,t,u) + O(\epsilon^2) \tag{3}$$

$$x^* = x + \epsilon\xi(x,t,u) + O(\epsilon^2) \tag{4}$$

$$t^* = t + \epsilon\tau(x,t,u) + O(\epsilon^2). \tag{5}$$

The invariance of equation (1) under the infinitesimal transformations (3)–(5) and the fact that the derivatives of u are independent leads to a set of determining equations. These determining equations are linear partial differential equations (PDEs) in η, ξ and τ. They are given by

$$\xi_u = 0 \qquad \tau_u = \tau_x = 0 \tag{6}$$

$$D\eta_{uu} + D'\eta_u + \left(D'' - \frac{D'}{D}\right)\eta = 0 \tag{7}$$

$$2\xi_x + \left(\frac{f'}{f} - \frac{g'}{g}\right)\xi - \eta\left(\frac{D'}{D}\right) - \tau_t = 0 \tag{8}$$

$$Dg\eta_{xx} + (Dg' - k')\eta_x - f\eta_t = 0 \tag{9}$$

and

$$Dg''\xi + (D'k' - k''D)\frac{\eta}{D} + (Dg' - k')\xi_x + 2D'g\eta_x - Dg\xi_{xx} + f\xi_t$$
$$- \frac{g'}{g}(Dg' - k')\xi + 2Dg\eta_{xu} = 0. \tag{10}$$

The solutions of the system of equations (6)–(10) lead to the group classification of equation (1) which was found to depend on whether the product $f(x)g(x)$ is a constant or not (in fact, equation (2) may give an indication of the fact that the group classification of equation (1) depends on whether the product of f and g equals a constant or not), which leads to two different cases (i) and (ii) as follows:

(i) $f(x)g(x) = 1$

$$\tau = (2c - A_0)t + \tau_0 \tag{11}$$

$$\xi = g(x)\left[c\int \frac{\mathrm{d}x}{g(x)} + \beta_1 t + \beta_0\right] \tag{12}$$

and

$$\eta = A_1 u + A_2 \tag{13}$$

where the functions $k(u)$ and $D(u)$ satisfy the remaining conditions

$$(A_1 u + A_2)k'' + (c - A_0)k' - \beta_1 = 0 \tag{14}$$

and

$$\left(\frac{A_1}{A_0}u + \frac{A_2}{A_0}\right) = \frac{D}{D'} \tag{15}$$

where c, A_0, A_1, A_2, τ_0, β_0 and β_1 are constants to be determined according to the different forms of the functions $g(x)$, $D(u)$ and $K(u)$.

(ii) $f(x)g(x) \neq 1$

$$\tau = at + \tau_0 \tag{16}$$

$$\xi = \sqrt{\frac{g(x)}{f(x)}}\left[\frac{1}{2}(A_0 + a)\int \sqrt{\frac{f(x)}{g(x)}}\,\mathrm{d}x + \beta_0\right] \tag{17}$$

and

$$\eta = A_1 u + A_2 \tag{18}$$

where the functions $g(x)$, $k(u)$ and $D(u)$ satisfy the remaining conditions

$$(A_1 u + A_2)k'' + (c - A_0)k' = 0 \tag{19}$$

$$\left(\frac{A_1}{A_0}u + \frac{A_2}{A_0}\right) = \frac{D}{D'} \tag{20}$$

and

$$\xi_x - \frac{g'(x)}{g(x)}\xi = c \tag{21}$$

where a is a constant to be determined according to the different forms of the functions $f(x)$, $g(x)$, $D(u)$ and $K(u)$. In fact, equations (14), (15), (19), (20) can easily be solved and the different forms of $D(u)$ and $K(u)$ can be found. Also equation (21) should be compatible with the expression of ξ in equation (17).

Now, it is clear that, according to our analysis, we have obtained the infinitesimals ξ, τ and η which are expressed in terms of the functions $f(x)$, $g(x)$, $D(u)$ and $K(u)$.

Different classes of (VCNDC) equation are considered and their corresponding infinitesimals (ξ, τ and η) are listed in tables 1–3. The results shown in tables 1–3 are found to be totally new. Also, it should be noted that putting $f(x)$ and $g(x)$ equal to 1 in table 3 will lead to all the results obtained by Edwards (except for the case $D(u) = \text{const}$) [2]. Also,

Table 1. Lie point symmetries for $f(x) = x^p$ and $g(x) = x^q$ where $p + q \neq 0$ and $p, q \in R$.

$D(u)$	$k(u)$	τ	ξ	η
u^m	$u^n, n \neq 1$	$A\left[\frac{(m-n+1)}{1-q}\nu - m\right]t + \tau_0$	$A\frac{(m-n+1)}{1-q}x$	Au
e^{mu}	e^{nu}	$A\left[\frac{(m-n)}{1-q}\nu - m\right]t + \tau_0$	$A\frac{(m-n)}{1-q}x$	A
u^m	$\ln(u)$	$A\left[\frac{(m+1)}{1-q}\nu - m\right]t + \tau_0$	$A\frac{(m+1)}{1-q}x$	Au
u^m	$u\ln(u) - u$	τ_0	0	0
e^{mu}	u^2	τ_0	0	0

A and τ_0 are the group constants and $\nu = p - q + 2, m \neq 0$.

Table 2. Lie point symmetries for $f(x) = e^{px}$ and $g(x) = e^{qx}$ where $p + q \neq 0$ and $p, q \in R$.

$D(u)$	$k(u)$	τ	ξ	η
u^m	$u^n, n \neq 1$	$A[(m-n+1)\delta - m]t + \tau_0$	$-A\frac{(m-n+1)}{q}x$	Au
e^{mu}	e^{nu}	$A[(m-n)\delta - m]t + \tau_0$	$-A\frac{(m-n)}{q}x$	A
u^m	$\ln(u)$	$A[(m+1)\delta - m]t + \tau_0$	$-A\frac{(m+1)}{q}x$	Au
u^m	$u\ln(u) - u$	0	0	0
e^{mu}	u^2	0	0	0

A and τ_0 are the group constants and $\delta = 1 - \frac{p}{q}, m \neq 0$.

Table 3. Lie point symmetries for $f(x)g(x) = 1$.

$D(u)$	$k(u)$	τ	ξ	η
u^m	$u^n, n \neq 1$	$A(m-2n+2)t + \tau_0$	$A(m-n+1)g(x)\int \frac{dx}{g(x)} + \beta g(x)$	Au
e^{mu}	e^{nu}	$A(m-2n)t + \tau_0$	$A(m-n)g(x)\int \frac{dx}{g(x)} + \beta g(x)$	A
u^m	$\ln(u)$	$A(m+2)t + \tau_0$	$A(m+1)g(x)\int \frac{dx}{g(x)} + \beta g(x)$	Au
u^m	$u\ln(u) - u$	$Amt + \tau_0$	$Amg(x)\int \frac{dx}{g(x)} + (At + \beta)g(x)$	Au
e^{mu}	u^2	$Amt + \tau_0$	$Amg(x)\int \frac{dx}{g(x)} + (2A + \beta)g(x)$	Au

A, τ_0 and β are the group constants and $m \neq 0$.

putting $f(x)$ and $g(x)$ equal to 1 in case (i) will lead directly to the group of well-known nonlinear diffusion–convection equations

$$u_t = (D(u)u_x)_x - k'(u)u_x \qquad \text{where} \quad D(u) \neq \text{const.} \tag{22}$$

Hence different and new classes of equation (22) could be considered. It was shown in [1, 2] that equation (22) has travelling-wave solutions, corresponding to the symmetry generator $\Gamma = \partial_t + c\partial_x$ where c is the speed of the wave, when $k(u)$ is either a power or exponential function. Also, putting $f(x)g(x) \neq$ const and $k(u) =$ const in case (ii), the group for the inhomogeneous nonlinear diffusion equation,

$$f(x)u_t = (g(x)D(u)u_x)_x$$

is obtained.

3. Exact solutions for the VCNDC equation

Due to (i) and (ii) different classes of the variable coefficient nonlinear diffusion–convection equation could be studied. Seven cases will be studied.

Case 1. Taking $f(x) = x^p$, $g(x) = x^q$, $D(u) = u^m$ and $k(u) = \text{const}$, where $m \neq 0$ and $p + q \neq 0$, equation (1) reduces to the inhomogeneous nonlinear diffusion equation

$$x^p u_t = (x^q u^m u_x)_x. \tag{23}$$

Broadbridge *et al* [11] have carried out a symmetry classification of a closely related equation to equation (23). Equation (23) may be rewritten as

$$\rho(x) u_t = x^{-q}(x^q u^m u_x)_x \tag{24}$$

where equation (23) describes, in appropriately normalized units, the propagation of a thermal wave in an inhomogeneous medium [5]. $\rho(x)$ is the particle density of the medium and $q = 0, 1, 2$ for the case of plane, axial and spherical symmetry, respectively. Some exact solutions for equation (24) have been constructed in [12].

According to our analysis, case (ii), equation (23) admits three infinitesimal generators, namely,

$$\Gamma_1 = \partial_t \qquad \Gamma_2 = \frac{-m}{\vartheta} x \partial_x - u \partial_u$$

and

$$\Gamma_3 = t\partial_t + \frac{x}{\vartheta}\partial_x \tag{25}$$

where $\vartheta = p - q + 2$.

The similarity variable ζ and similarity solution $F(\zeta)$ corresponding to the infinitesimal generator Γ_2 are

$$\zeta = t \qquad u(x,t) = x^{\frac{\vartheta}{m}} F(\zeta) \tag{26}$$

where $F(\zeta)$ satisfies the reduced ODE

$$F' = \left[\frac{\vartheta^2 + m\vartheta\,(p+1)}{m^2}\right] F^{m+1}. \tag{27}$$

Integrating equation (27) and using equation (26), we obtain

$$u(x,t) = R_0 x^{\frac{\vartheta}{m}} (R_1 - t)^{\frac{-1}{m}} \tag{28}$$

where R_0 and R_1 are constants.

Case 2. Putting $f(x) = \mathrm{e}^{px}$, $g(x) = \mathrm{e}^{qx}$, $D(u) = u^m$ and $k(u) = \text{const}$, where $p + q \neq 0$, in equation (1) we get the inhomogeneous nonlinear diffusion equation

$$\mathrm{e}^{px} u_t = (\mathrm{e}^{qx} u^m u_x)_x \tag{29}$$

which describes another version of the inhomogeneous nonlinear diffusion equation, the propagation of a thermal wave in an exponential atmosphere [5]. Equation (29) admits three infinitesimal generators, according to our analysis in (ii), namely,

$$\Gamma_1 = \partial_t \qquad \Gamma_2 = t\partial_t + \frac{1}{2\alpha}\partial_x$$

and

$$\Gamma_3 = \frac{1}{2\alpha}\partial_x + \frac{u}{m}\partial_u \tag{30}$$

where $\alpha = \frac{p-q}{2}$.

The similarity variable ζ and similarity solution $F(\zeta)$ corresponding to the infinitesimal generator Γ_3 are

$$\zeta = t \qquad u(x,t) = \mathrm{e}^{\frac{2\alpha x}{m}} F(\zeta) \tag{31}$$

where $F(\zeta)$ satisfies the reduced ODE

$$F' = 4\left(\frac{\alpha}{m}\right)^2\left[1 + m\left(1 + \frac{q}{2\alpha}\right)\right]F^{m+1}. \tag{32}$$

Integrating equation (32) and using equation (31), we obtain

$$u(x,t) = \mathrm{e}^{\frac{2\alpha x}{m}}\,(R_0 - R_1 t)^{\frac{-1}{m}} \tag{33}$$

where R_0 and R_1 are constants.

Case 3. Putting $f(x) = g(x) = 1$, $D(u) = \mathrm{e}^{mu}$ and $k(u) = \mathrm{e}^{nu}$ in equation (1) we obtain

$$u_t = (\mathrm{e}^{mu}u_x)_x - n\,\mathrm{e}^{nu}u_x. \tag{34}$$

Equation (34) applies to unsaturated flow in porous media [4] which admits, according to table 3, the infinitesimal generators

$$\Gamma_1 = \partial_t \qquad \Gamma_2 = \partial_x$$

and

$$\Gamma_3 = (m-n)x\partial_x + (m-2n)t\partial_t + \partial_u. \tag{35}$$

The similarity variable ζ and similarity solution $F(\zeta)$ corresponding to the infinitesimal generator Γ_3 are

$$\zeta = xt^{-\frac{(m-n)}{(m-2n)}} \qquad u(x,t) = F(\zeta) + \frac{1}{(m-2n)}\ln(t) \tag{36}$$

where $n \neq \frac{m}{2}$ and $F(\zeta)$ satisfies the reduced ODE (table 3 of [2])

$$\frac{1}{(m-2n)}[1 - (m-n)\zeta F'] = \frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\mathrm{e}^{mF}F' - \mathrm{e}^{nF}\right] \tag{37}$$

which can be integrated for the case $n = m$ and using equation (36) we obtain

$$u(x,t) = x + \frac{1}{m}\ln\left[\frac{\mathrm{e}^{-mx}\,(1 + m(c_1 m + x) + c_2\,\mathrm{e}^{mx})}{m^2 t}\right] \tag{38}$$

where c_1 and c_2 are constants.

For the case $n = \frac{m}{2}$, the invariants of the group corresponding to the generator Γ_3 are

$$\zeta = x \qquad u(x,t) = F(\zeta) + \frac{2}{m}\ln(t) \tag{39}$$

where $F(\zeta)$ satisfies the reduced ODE

$$F' = -\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\mathrm{e}^{\frac{m}{2}F}F' - \frac{2}{m}\,\mathrm{e}^{mF}\right]. \tag{40}$$

Integrating equation (40) and using equation (39) will result in a solution, in an implicit form, for $u(x,t)$ which reads

$$m\left(-mt + 2\,\mathrm{e}^{-\frac{m}{2}u}x\right) + 4\ln\left[-2 + m\,\mathrm{e}^{-\frac{m}{2}u}x\right] = c_1 m^3 \tag{41}$$

where c_1 is a constant.

Case 4. Taking $f(x) = x^{-q}$, $g(x) = x^q$, $D(u) = u^m$ and $k(u) = u$, where $q \neq (0, 1)$ and $m \neq 0$, equation (1) becomes

$$x^{-q}u_t = (x^q u^m u_x)_x - u_x \tag{42}$$

where it admits, according to our analysis in (i), the infinitesimal generators

$$\Gamma_1 = \partial_t \qquad \Gamma_2 = x^q\partial_x \qquad \Gamma_3 = \frac{1}{2}\left(\frac{x}{(1-q)} + tx^q\right)\partial_x + t\partial_t$$

and

$$\Gamma_4 = \frac{m}{2}\left(\frac{x}{(1-q)} - tx^q\right)\partial_x + u\partial_u. \tag{43}$$

The similarity variable ζ and similarity solution $F(\zeta)$ corresponding to the infinitesimal generator Γ_4 are

$$\zeta = t \qquad u(x,t) = F(\zeta)x^{\frac{-2q}{m}}(x + (q-1)tx^q)^{\frac{2}{m}} \tag{44}$$

where $F(\zeta)$ satisfies the reduced ODE

$$F' = \frac{2}{m^2}(2+m)(q-1)^2 F^{m+1}. \tag{45}$$

Integrating equation (45) and using equation (44) will result in a solution for $u(x,t)$ which reads

$$u(x,t) = 2^{\frac{-1}{m}} x^{\frac{-2q}{m}}(x + (q-1)tx^q)^{\frac{2}{m}}\left[\frac{m}{(2+m)(q-1)^2(c_0 - t)}\right]^{\frac{1}{m}} \tag{46}$$

where c_0 is a constant.

Case 5. Putting $f(x) = x^{-1}$, $g(x) = x$, $D(u) = u^m$ and $k(u) = u$, where $m \neq 0$, in equation (1) reduces it to

$$x^{-1}u_t = (xu^m u_x)_x - u_x \tag{47}$$

where it admits, according to our analysis in (i), the infinitesimal generators

$$\Gamma_1 = \partial_t \qquad \Gamma_2 = x\partial_x \qquad \Gamma_3 = \frac{x}{2}(\ln(x) + t)\,\partial_x + t\partial_t$$

and

$$\Gamma_4 = \frac{mx}{2}(\ln(x) - t)\,\partial_x + u\partial_u. \tag{48}$$

The similarity variable ζ and similarity solution $F(\zeta)$ corresponding to the infinitesimal generator Γ_4 are

$$\zeta = t \qquad u(x,t) = F(\zeta)\,(-t + \ln(x))^{\frac{2}{m}} \tag{49}$$

where $F(\zeta)$ satisfies the reduced ODE

$$F' = \frac{2}{m^2}(2+m)F^{m+1}. \tag{50}$$

Integrating equation (50) and using equation (49) will result in a solution for $u(x,t)$ which reads

$$u(x,t) = 2^{\frac{-1}{m}}(-t + \ln(x))^{\frac{2}{m}}\left[\frac{m}{(2+m)(c_0 - t)}\right]^{\frac{1}{m}} \tag{51}$$

where c_0 is a constant.

Case 6. Taking $f(x) = e^{-qx}$, $g(x) = e^{qx}$, $D(u) = u^m$ and $k(u) = u$, where $q \neq 0$ and $m \neq 0$, equation (1) becomes

$$e^{-qx}u_t = (e^{qx}u^m u_x)_x - u_x \tag{52}$$

where it admits, according to our analysis in (i), the infinitesimal generators

$$\Gamma_1 = \partial_t \qquad \Gamma_2 = e^{qx}\partial_x \qquad \Gamma_3 = \frac{1}{2q}(qt\,e^{qx} - 1)\partial_x + t\partial_t$$

and

$$\Gamma_4 = \frac{-m}{2q}(qt\,\mathrm{e}^{qx} - 1)\partial_x + u\partial_u. \tag{53}$$

The similarity variable ζ and similarity solution $F(\zeta)$ corresponding to the infinitesimal generator Γ_4 are

$$\zeta = t \qquad u(x,t) = F(\zeta)\,\mathrm{e}^{-\frac{2qx - 2\ln[qt\mathrm{e}^{qx}+1]}{m}} \tag{54}$$

where $F(\zeta)$ satisfies the reduced ODE

$$F' = \frac{2}{m^2}(2+m)q^2F^{m+1}. \tag{55}$$

Integrating equation (55) and using equation (54) will result in a solution for $u(x,t)$ which reads

$$u(x,t) = 2^{\frac{-1}{m}}\,\mathrm{e}^{-\frac{2qx}{m}}(1 + qt\,\mathrm{e}^{qx})^{\frac{2}{m}}\left[\frac{m}{(2+m)q^2(c_0 - t)}\right]^{\frac{1}{m}} \tag{56}$$

where c_0 is a constant.

Case 7. Taking $f(x) = x$, $g(x) = x$, $D(u) = u^n$ and $k(u) = \frac{\mu+1}{n+1}u^{n+1}$, where $n \neq (0, -1)$ and $m \neq -1$, equation (1) reduces to

$$xu_t = (xu^nu_x)_x - (\mu+1)u^nu_x. \tag{57}$$

Equation (57) admits, according to our analysis in (ii), the infinitesimal generators

$$\Gamma_1 = \partial_t \qquad \Gamma_2 = 2t\partial_t + x\partial_x$$

and

$$\Gamma_3 = nx\partial_x + 2u\partial_u. \tag{58}$$

The similarity variable ζ and similarity solution $F(\zeta)$ corresponding to the infinitesimal generator Γ_3 are

$$\zeta = t \qquad u(x,t) = x^{\frac{2}{n}}F(\zeta) \tag{59}$$

where $F(\zeta)$ satisfies the reduced ODE

$$F' = \frac{2}{n^2}(2 - n(\mu-1))\,F^{n+1}. \tag{60}$$

Integrating equation (60) and using equation (59), we obtain

$$u(x,t) = \left[\frac{nx^2}{2\,(2 - n(\mu-1))\,(c_0 - t)}\right]^{\frac{1}{n}} \tag{61}$$

where c_0 is a constant.

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